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Skyrme Crystal from a Twisted Instanton on a Four-Torus

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Abstract

We describe how an approximation to the minimal energy Skyrme crystal can be obtained from the holonomy of a Yang-Mills instanton. The appropriate instanton is twisted on a four-torus and has instanton number equal to one half. It generates a Skyrme field with the correct topological and symmetry properties of the crystal. An explicit solution for the instanton is not known, but an analytical fit to numerical data is available and using this we obtain a Skyrme crystal whose energy is only 2% above that of the (numerically) known solution.

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1 Introduction

The Skyrme model is an effective theory of low energy hadron physics in which nucleons are described by topological solitons. The baryon number B is identified with the topological charge (or number of solitons) and it is of interest to determine the classical soliton solutions, since they describe the classical state of the nucleons. However, the Skyrme field equation is a nonlinear partial differential equation which is difficult to solve. In fact, even for the simplest case of $B = 1$ the minimal energy solution is only known numerically.

In a region with high baryon number, the solitons form a crystal. Klebanov [1] began the study of Skyrme crystals and over the last 10 years various numerical and analytical techniques have been applied to find the solution with minimum energy per baryon.

It has proved useful for understanding the solitons of the Skyrme model to construct approximate solutions by calculating the holonomy of Yang-Mills instantons [2]. This method has led to a good understanding of the geometry of the $B = 2$ sector [3], and the tetrahedral and cubic symmetries of the minimal energy $B = 3$ and $B = 4$ solutions respectively [4]. The instanton method has computational advantages, and it provides a means by which the full soliton theory may be truncated to a finite number of degrees of freedom. However, one area in which it has yet to make a contribution is that of the Skyrme crystal. In this letter we take the first steps in applying the instanton method to the Skyrme crystal by identifying a suitable Yang-Mills instanton. We also make a qualitative calculation based on numerical data for the instanton.

2 Skyrme Crystal

The field $U(\mathbf{x})$ of the static Skyrme model is an $SU(2)$ matrix-valued function of the space coordinates $\mathbf{x} = (x_1, x_2, x_3) = (x, y, z)$. It may be expressed in terms of sigma and pion fields as

$$U = \sigma + i\boldsymbol{\pi} \cdot \boldsymbol{\tau} \tag{1}$$

where $\boldsymbol{\pi} = (\pi_1, \pi_2, \pi_3)$ is a triplet of pion fields and $\boldsymbol{\tau}$ denotes the Pauli matrices. The normalization constraint is $\sigma^2 + \boldsymbol{\pi}.\boldsymbol{\pi} = 1$. For finite energy

fields, it is required that as $|\mathbf{x}| \to \infty$, $U(\mathbf{x}) \to 1$. The energy density is

$$\mathcal{E} = -\frac{1}{24\pi^2}\mathrm{Tr}(R_i R_i) - \frac{1}{192\pi^2}\mathrm{Tr}([R_i, R_j][R_i, R_j]) \tag{2}$$

and the baryon density is

$$\mathcal{B} = -\frac{1}{24\pi^2}\epsilon_{ijk}\mathrm{Tr}(R_i R_j R_k) \tag{3}$$

where R_i is the right current $\partial_i U U^{-1}$. The spatial integrals of $\mathcal{E}$ and $\mathcal{B}$ give the energy E and the baryon number (topological charge) B of the Skyrme field. The above units are chosen so that the Fadeev-Bogomolny bound on the energy E is simply

$$E \geq |B|. \tag{4}$$

In flat space this bound cannot be attained (except by the vacuum solution $U = 1$) and the single Skyrmion ($B = 1$) solution has energy $E = 1.23$.

With certain relative orientations well separated Skyrmions attract, and at high density form a crystal. Klebanov's original crystal [1] was a simple cubic lattice of Skyrmions whose symmetries were motivated by the attempt to rotate the Skyrmions so as to maximize the attraction between nearest neighbours. It has energy per baryon of $E = 1.08$. Other symmetries were proposed which lead to slightly lower, but not minimal energy crystals [5]. Following the work of Castillejo et al. [6], and of Kugler and Shtrikman [7], it is now understood that it is best to arrange the Skyrmions initially as a face-centred cubic lattice. Their orientations can be chosen very symmetrically to give maximal attraction between all nearest neighbours. This configuration relaxes to the minimal energy crystal, the Skyrme crystal, and in the process the symmetry increases further. The numerical investigations of [6], and the variational approach of [7] using fourier series, lead to the conclusion that the Skyrme crystal has energy per baryon of $E = 1.038$ and a cubic unit cell of side $L = 4.7$. It is a crystal of half-Skyrmions, as a unit cell contains half a unit of baryon number. The fields are strictly periodic after translation by $2L$ in the x,y or z directions, and within a cube of side $2L$ the baryon number is $B = 4$. The crystal symmetries, which differ from those of the crystals proposed in refs. [1] and [5], involve translations, reflections and rotations of the space coordinates combined with O(4) rotations of the sigma and pion

fields. Explicitly, the symmetry generators are [7]

$$\begin{aligned}
(x,y,z) &\to (-x,y,z) && \& \quad (\sigma,\pi_1,\pi_2,\pi_3) \to (\sigma,-\pi_1,\pi_2,\pi_3) && (5)\\
(x,y,z) &\to (y,z,x) && \& \quad (\sigma,\pi_1,\pi_2,\pi_3) \to (\sigma,\pi_2,\pi_3,\pi_1) && (6)\\
(x,y,z) &\to (x,z,-y) && \& \quad (\sigma,\pi_1,\pi_2,\pi_3) \to (\sigma,\pi_1,\pi_3,-\pi_2) && (7)\\
(x,y,z) &\to (x+L,y,z) && \& \quad (\sigma,\pi_1,\pi_2,\pi_3) \to (-\sigma,-\pi_1,\pi_2,\pi_3). && (8)
\end{aligned}$$

The fields obtained numerically, or by optimising the fourier series, are very well approximated by the analytical formulae [6]

$$\begin{aligned}
\sigma &= c_1 c_2 c_3 \\
\pi_1 &= -s_1\sqrt{1-\frac{1}{2}s_2^2-\frac{1}{2}s_3^2+\frac{1}{3}s_2^2 s_3^2} \qquad \text{and cyclic}
\end{aligned} \tag{9}$$

where

$$s_i = \sin(\frac{\pi x_i}{L}) \quad \text{and} \quad c_i = \cos(\frac{\pi x_i}{L}). \tag{10}$$

3 Twisted Instantons on T^4

The pioneering work on Yang-Mills instantons on a four-torus T^4 with twisted boundary conditions was done by 't Hooft [8], motivated by the study of quark confinement in QCD. Let A_μ be the $su(2)$-valued Yang-Mills gauge potential, with field strength $F_{\mu\nu}$. Here greek indices take the values 0 to 3, and latin indices 1 to 3, and we write $x_\mu = (t, x, y, z)$, with t the euclidean time coordinate. We model the four-torus by the euclidean box defined by $0 \le x_\mu \le a_\mu$. To give the boundary conditions we introduce $SU(2)$-valued twist matrices Ω_μ each of which is independent of the μth spacetime coordinate. The twisted boundary conditions are that the gauge potentials are periodic modulo gauge transformations

$$A_\nu(x_\mu = a_\mu) = \Omega_\mu^{-1} A_\nu(x_\mu = 0)\Omega_\mu + \Omega_\mu^{-1}\partial_\nu\Omega_\mu. \tag{11}$$

The notation $A_\nu(x_\mu = a_\mu)$ means that the μth coordinate is set equal to a_μ but the remaining coordinates are arbitrary. On 2-faces of the box the compatibility conditions which arise from (11) are

$$\Omega_\mu(x_\nu = a_\nu)\Omega_\nu(x_\mu = a_\mu) = \Omega_\nu(x_\mu = a_\mu)\Omega_\mu(x_\nu = a_\nu)Z_{\mu\nu} \tag{12}$$

where each $Z_{\mu\nu}$ is an element of the centre $\mathbb{Z}_2$ of $SU(2)$. We introduce the gauge invariant antisymmetric twist integers $n_{\mu\nu} \in \mathbb{Z}$ (mod 2) by

$$Z_{\mu\nu} = \exp(i\pi n_{\mu\nu}). \tag{13}$$

It is these that are the important quantities, since any twist matrices with the same set of twist integers are gauge equivalent. It is convenient to collect the twist integers into a magnetic flux vector $\mathbf{m}$ and an electric flux vector $\mathbf{k}$, where $n_{ij} = \epsilon_{ijk} m_k$ and $k_i = n_{0i}$.

't Hooft pointed out that the usual expression for the instanton number does not have to be integer valued in the presence of twist. In fact

$$N = \frac{1}{16\pi^2} \int_{T^4} \mathrm{Tr}(*F_{\mu\nu} F^{\mu\nu}) d^4x = q + \frac{\mathbf{k} \cdot \mathbf{m}}{2} \tag{14}$$

where q is an integer. The mathematical explanation of this result can be found in [9], [10]. The expression (14) is minus the second Chern number for an $SU(2)$ gauge theory, but since the gauge functions are only defined modulo the centre of the gauge group, we are in fact dealing with the gauge group $SU(2)/\mathbb{Z}_2 = SO(3)$, and that is why N can be an integer or half-integer.

Many elegant mathematical results are known for twisted instantons on T^4, such as an existence proof [11], a (non-explicit) construction of the appropriate bundles in terms of K-theory [10] and a Nahm duality transformation [12]. The moduli (parameter) space of instantons of instanton number N is $8N$, that is, an integer multiple of four [13].

4 Skyrme Crystal from an Instanton

We now show how an approximation to the Skyrme crystal described in section 2 can be obtained from the holonomy of a suitable instanton on T^4.

In order to obtain a Skyrme field with the correct cubic symmetry we set $a_i = L$ and $a_0 = T$, and choose isotropic magnetic and electric flux vectors $\mathbf{m} = (1,1,1)$ and $\mathbf{k} = (1,1,1)$. Then (14) becomes $N = q + \frac{3}{2}$. There are explicit abelian solutions [8] for $N = \frac{3}{2}$, when the four-torus has the same length in each direction, but we are interested in approximating a crystal composed of half-Skyrmions so we wish to consider the case $N = \frac{1}{2}$. Note that the Yang-Mills action on T^4 is bounded by $8\pi^2|N|$, so the $N = \frac{1}{2}$ instantons are the minimal action configurations outside the vacuum sector.

Moreover, their moduli space is four-dimensional and simply parametrises the translates, and certain discrete gauge transformations of, a unique $N = \frac{1}{2}$ instanton on the given torus. This instanton is invariant under the cubic subgroup of the spatial rotation group, and may be assumed to be centred with its maximum action density at $x = y = z = L/2$ and $t = T/2$.

By choosing explicit twist matrices we perform a partial gauge fixing. Here it is convenient to use the Pauli matrices for the spatial twist matrices

$$\Omega_j = i\tau_j. \tag{15}$$

It is a simple matter to check that this choice satisfies the spatial components of (12) with the twist integers given earlier. It is also easy to construct a twist matrix Ω_0 which satisfies the remaining components of (12). Although we shall not need an explicit form for this matrix we give a possible choice for completeness,

$$\Omega_0 = (\prod_{i=1}^{3} c_i) + i\sqrt{\frac{1-(\prod_{i=1}^{3} c_i)^2}{\mathbf{c}\cdot\mathbf{c}}}\mathbf{c}\cdot\boldsymbol{\tau} \tag{16}$$

where $c_i = \cos(\pi x_i/L)$.

The Skyrme field generated by the instanton is, by definition, the holonomy of the instanton along all lines parallel to the euclidean time axis, *ie*

$$U(\mathbf{x}) = \left(\mathcal{P}\exp\int_0^T A_0(x_0,\mathbf{x})\ dx_0\right)\Omega_0(\mathbf{x}) \tag{17}$$

where $\mathcal{P}$ denotes path ordering. Note the inclusion of the twist matrix since the path must be a closed loop. The Skyrme field is defined on a cube of side L, and is extended by symmetry and continuity to a crystal with this cube as unit cell. The usual argument for instantons on $\mathbb{R}^4$ can be repeated in this case to show that the instanton number N equals the baryon number B of the Skyrme field that it generates. Hence the Skyrme field has $B = \frac{1}{2}$ in a unit cell.

We can show, without using the explicit form (16), that the Skyrme field has the symmetry (8) of a half-Skyrmion crystal. Using (11) and (15) we have

$$U(x+L,y,z) = \tau_1\left(\mathcal{P}\exp\int_0^T A_0(x_0,x,y,z)\ dx_0\right)\tau_1\ \Omega_0(x+L,y,z). \tag{18}$$

Equation (12) implies that

$$\Omega_0(x+L,y,z) = -\tau_1\Omega_0(x,y,z)\tau_1, \tag{19}$$

and combining (18) and (19) gives

$$U(x+L,y,z) = -\tau_1 U(x,y,z)\tau_1, \tag{20}$$

which is equivalent to symmetry (8). The symmetries (5)-(7) are a consequence of the cubic symmetry of the instanton.

To progress, we would like to have an explicit analytical solution for the instanton of interest, but this is not available. Fortunately some numerical studies have been performed by García Pérez et al. [14] and we can make use of their results as a first step in determining the instanton-generated Skyrme crystal, and estimating how good an approximation it is. García Pérez et al. treat the Yang-Mills theory as a lattice gauge theory and search for the minimal action configuration using the lattice cooling method. For the case $T=\infty$ they provide an analytical fit to the numerical data for the holonomy, and it is this result which we shall use here. Although $T=\infty$ the instanton is still localized in the euclidean time direction, with a scale which is determined by the spatial length of the four-torus, L. (In this limit, one lets the range of t be $-\infty < t < \infty$ and assumes that the instanton has its maximal action density at $t=0$.) The analytical fit from [14] for the holonomy in the euclidean time direction is

$$U(\mathbf{x}) = (\prod_{i=1}^{3} h_i) + i\sqrt{\frac{1-(\prod_{i=1}^{3} h_i)^2}{\mathbf{f}\cdot\mathbf{f}}}\mathbf{f}\cdot\boldsymbol{\tau} \tag{21}$$

where

$$h_i = \cos(\frac{\pi x_i}{L})(1+\alpha\sin^2(\frac{\pi x_i}{L})), \tag{22}$$

$$f_i = -\sin(\frac{\pi x_i}{L})(1+\beta\cos^2(\frac{\pi x_i}{L})). \tag{23}$$

Here $\alpha = -0.196$ and $\beta = -0.172$ are constants whose values are determined by the fit. Note that this Skyrme field has a similar form to the approximation to the Skyrme crystal (9) mentioned earlier. It is a simple matter to check explicitly that it has all the symmetry properties (5) to (8) of the Skyrme crystal.

It is relatively easy to obtain highly accurate numerical integrals for quantities such as the baryon number and energy of the Skyrme field (21) since the differentiation of U can be performed analytically, and furthermore the region of integration is finite. Integrating the baryon density (3) over a cube of side L confirms the baryon number to be $B = \frac{1}{2}$, and we find the energy per baryon is

$$E = 0.1069L + \frac{2.6146}{L}. \tag{24}$$

Hence the minimum energy is $E = 1.058$ at $L = 4.9$, with both these figures being reasonably close to the Skyrme crystal values. One may take the view that the crystal size should be fixed at the value $L = 4.7$. The approximation (21) then contains no free parameters at the expense of an energy increase of 0.1%. The fact that the energy is only 2% above that of the Skyrme crystal is encouraging, given that we are using only a fit to numerical data. Errors will be introduced both in the numerical construction of the instanton and in the analytical fitting process, and it would be very interesting if a more accurate numerical method could be employed in calculating the instanton and its holonomy.

In [14] it is noted that moderate variations of the parameters α and β may not significantly alter the quality of the fit to the numerical data. To see whether a more accurate scheme may produce a better approximate Skyrme crystal, we allow α and β to be arbitrary parameters. Using a steepest decent algorithm in (α, β) space we find that the minimum crystal energy is obtained when $\alpha = -0.02$ and $\beta = -0.25$. At these values the energy is $E = 1.040$ for a crystal size $L = 4.7$, *ie* the energy is within $\frac{1}{5}\%$ of the true minimum.

Allowing T to vary could produce a better approximation to the Skyrme crystal. In [15] it was shown that a slightly better approximation to the $B = 1$ Skyrmion in $\mathbb{R}^3$ can be obtained from an instanton on $\mathbb{R}^3$ times a finite euclidean time interval (a caloron) than from an instanton on $\mathbb{R}^4$. Moreover, a lower dimensional analogue of the procedure described in this letter exists [16], where sine-Gordon kink chains are approximated by the holonomies of $\mathbb{CP}^1$ instantons on T^2. In that example a much better approximation is obtained by having a finite length for the torus in the euclidean time direction. If this analogue result is a good guide to what happens in the case of the Skyrme crystal then one must certainly consider finite T.

5 Conclusion

In this letter we have identified the Yang-Mills instanton whose holonomy produces a Skyrme field which has the topological and symmetry properties of the Skyrme crystal. Since an explicit solution is not known for the instanton it is not an easy task to obtain accurate quantitative information about the generated crystal. However, by using numerical work of García Pérez et al. [14] we have obtained some initial results which are encouraging. We hope that a more detailed investigation of this problem will prove fruitful.

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